MATH1010 Assignment 4

Suggested Solution

1. the degree-6 Taylor polynomial of the function f about the point c :

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(6)}(c)}{6!}(x-c)^6$$

(a) $21+(-36)(x+2)+\frac{50}{2!}(x+2)^2+\frac{(-48)}{3!}(x+2)^3+\frac{24}{4!}(x+2)^4$

(b) $3x+\frac{-27}{3!}x^3+\frac{243}{5!}x^5$

(c) $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}(x-\frac{\pi}{4})+\frac{-\frac{1}{\sqrt{2}}}{2!}(x-\frac{\pi}{4})^2+\frac{\frac{1}{\sqrt{2}}}{3!}(x-\frac{\pi}{4})^3+\frac{\frac{1}{\sqrt{2}}}{4!}(x-\frac{\pi}{4})^4+\frac{-\frac{1}{\sqrt{2}}}{5!}(x-\frac{\pi}{4})^5+\frac{-\frac{1}{\sqrt{2}}}{6!}(x-\frac{\pi}{4})^6$

(d) $6+3x+\frac{27}{2!}x^2+\frac{51}{3!}x^3+\frac{291}{4!}x^4+\frac{963}{5!}x^5+\frac{4227}{6!}x^6$

(e) $\sqrt{3}+\frac{\sqrt{3}}{6}(x-1)+\frac{-\frac{\sqrt{3}}{36}}{2!}(x-1)^2+\frac{\frac{\sqrt{3}}{72}}{3!}(x-3)^3+\frac{-\frac{5\sqrt{3}}{432}}{4!}(x-1)^4+\frac{\frac{35\sqrt{3}}{2592}}{5!}(x-1)^5+\frac{-\frac{35\sqrt{3}}{1728}}{6!}(x-1)^6$

(f) $\sqrt[3]{5}+\frac{2\sqrt[3]{5}}{15}(x-2)+\frac{-\frac{8\sqrt[3]{5}}{225}}{2!}(x-2)^2+\frac{\frac{16\sqrt[3]{5}}{675}}{3!}(x-2)^3+\frac{-\frac{256\sqrt[3]{5}}{10125}}{4!}(x-2)^4+\frac{\frac{5632\sqrt[3]{5}}{151875}}{5!}(x-2)^5+\frac{-\frac{157696\sqrt[3]{5}}{2278125}}{6!}(x-2)^6$

(g) $\frac{1}{2}+\frac{-\frac{9}{4}}{2!}x^2+\frac{\frac{243}{4}}{4!}x^4+\frac{-\frac{32805}{8}}{6!}x^6$

(h) $\frac{1}{4}-\frac{5}{16}(x-1)+\frac{\frac{5}{32}}{2!}(x-1)^2+\frac{-\frac{15}{128}}{3!}(x-1)^3+\frac{\frac{15}{128}}{4!}(x-1)^4+\frac{-\frac{75}{512}}{5!}(x-1)^5+\frac{\frac{225}{1024}}{6!}(x-1)^6$

(i) $1+e^{-1}t+\frac{-e^{-2}}{2!}t^2+\frac{2e^{-3}}{3!}t^3+\frac{-6e^{-4}}{4!}t^4+\frac{24e^{-5}}{5!}t^5+\frac{-120e^{-6}}{6!}t^6$

(j) $\ln(1-e^2t^2)=\ln(1-et)(1+et)=\ln(1-et)+\ln(1+et)$

$\frac{-2e^2}{2!}t^2+\frac{-12e^4}{4!}t^4+\frac{-240e^6}{6!}t^6$

(k) By simplification, $\cos 3x\cos 2x=\frac{1}{2}[\cos 5x+\cos x]$

$1+\frac{-13}{2!}x^2+\frac{313}{4!}x^4+\frac{-7813}{6!}x^6$

(l) By simplification, $\sin^2(4x) - \sin^2(3x) = \frac{1}{2}[\cos 6x - \cos 8x]$

$\frac{14}{2!}x^2 + \frac{-1400}{4!}x^4 + \frac{107744}{6!}x^6$

2. (a) let $f(x) = \sec x$

Since f is an even function, we have $f'(0) = f^{(3)}(0) = f^{(5)}(0) = 0$

$f''(x)\cos x + 2f'(x)(-\sin x) + \sec x(-\cos x) = 0$

$f''(0) = 1$

$f^{(4)}(x)\cos x + 4f^{(3)}(-\sin x) + 6f^{(2)}(x)(-\cos x) + 4f'(x)(\sin x) + \sec x(\cos x) = 0$

$f^{(4)}(0) = 5$

$f^{(6)}(x)\cos x + 6f^{(5)}(x)(-\sin x) + 15f^{(4)}(x)(-\cos x) + 20f^{(3)}(x)(\sin x) + 15f^{(2)}(x)(\cos x) + 6f'(x)(-\sin x) +$

$\sec x(-\cos x) = 0$

$f^{(6)}(0) = 61$

So the degree 6 Taylor polynomial of $\sec x$ centered at 0 is

$$1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6$$

(b) let $f(x) = \tan x$

Since f is an odd function, we have $f(0) = f^{(2)}(0) = f^{(4)}(0) = 0$

$f'(0) = 1$

$f^{(3)}(x)\cos x + 3f''(x)(-\sin x) + 3f'(x)(-\cos x) + f(x)(\sin x) = -\cos x$ and $f^{(3)}(0) = 2$

$f^{(5)}(x)\cos x + 5f^{(4)}(x)(-\sin x) + 10f^{(3)}(x)(-\cos x) + 10f''(x)(\sin x) + 5f'(x)\cos x + f(x)(-\sin x) =$

$\cos x$ and $f^{(5)}(0) = 16$

So the degree 6 Taylor polynomial of $\tan x$ centered at 0 is

$$x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5$$

3. Let $\alpha \in (0,1)$ and $x \in (0,1)$

Let $f(x) = (1+x)^\alpha$

(a) By Taylor's Theorem, we have

$$f(x) = f(0) + \sum_{n=1}^{N} f\frac{f^{(n)}(0)}{n!} + \frac{f^{(N+1)(c)}}{(N+1)!}x^{N+1}$$

$$= 1 + \sum_{n=1}^{N} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}$$

$$\left|(1+x)^\alpha - 1 - \sum_{n=1}^{N} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n\right| = \frac{\left|f^{(N+1)}(c)\right|}{(N+1)!}x^{N+1}$$

On the other hand,

$$f^{(N+1)}(c) = \alpha(\alpha-1)\dots(\alpha-N)(1+c)^{\alpha-(N+1)}$$

$$\frac{|f^{(N+1)}(c)|}{(N+1)!} = \frac{\alpha(1-\alpha)\dots(N-\alpha)(1+c)^{\alpha-(N+1)}}{1\dots N(N+1)}$$

$$\leqslant \frac{2\alpha}{N+1}$$

(b) For $N = 4$, we have $\frac{2 \cdot 0.2 \cdot 0.2^5}{5} \leqslant \frac{1}{10^4}$
Then $\left|(1.2)^{0.2} - 1 - \sum_{n=1}^{4} \frac{0.2(0.2-1)\dots(0.2-n+1)}{n!}x^n\right| \leqslant \frac{2(0.2)(0.2)^5}{5} \leqslant \frac{1}{10^4}$
Hence

4. (a)

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{\sin 3x}{3x}\frac{5x}{\sin 5x}\frac{3}{5}$$

$$= \frac{3}{5}$$

(b)

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} 1 + \cos x$$

$$= 2$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x} &= \lim_{x\to 0}\frac{2\cos x-2\cos 2x}{1-\cos x}\\ &= \lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{\sin x}\\ &= \lim_{x\to 0}-2+8\cos x\\ &= 6\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0}\frac{1-x\cot x}{x\sin x} &= \lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin^2 x}\\ &= \lim_{x\to 0}\frac{x}{\sin x+2x\cos x}\\ &= \lim_{x\to 0}\frac{1}{3\cos x-2x\sin x}\\ &= \frac{1}{3}\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh x-\sin x}{x(\cosh x-\cos x)} &= \lim_{x\to 0}\frac{\cosh x-\cos x}{\cosh x-\cos x+x(\sinh x+\sin x)}\\ &= \lim_{x\to 0}\frac{\sinh x+\sin x}{2\sinh x+2\sin x+x(\cosh x+\cos x)}\\ &= \lim_{x\to 0}\frac{\cosh x+\cos x}{3(\cosh x+\cos x)+x(\sinh x-\sin x)}\\ &= \frac{1}{3}\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to 0}\frac{\ln\cos 2x}{\ln\cos x} &= \lim_{x\to 0}\frac{2\sin 2x\cos x}{\cos 2x\sin x}\\ &= \lim_{x\to 0}\frac{2\tan 2x}{\tan x}\\ &= 4\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}-\frac{1}{e^x-1} &= \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}\\ &= \lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}\\ &= \lim_{x\to 0}\frac{e^x}{2e^x+xe^x}\\ &= \frac{1}{2}\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to 1}\frac{1}{\ln x}-\frac{1}{x-1} &= \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)(\ln x)}\\ &= \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}\\ &= \lim_{x\to 1}\frac{x-1}{x\ln x+x-1}\\ &= \lim_{x\to 1}\frac{1}{\ln x+2}\\ &= \frac{1}{2}\end{aligned}$$

(i)

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-x-1}{\cosh x-1} &= \lim_{x\to 0}\frac{e^x-1}{\sinh x}\\ &= \lim_{x\to 0}\frac{e^x}{\cosh x}\\ &= 1\end{aligned}$$

(j)

$$\lim_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x} = \lim_{x\to 0} -(1+x)^{\frac{1}{x}}[-\frac{\ln(1+x)}{x^2}+\frac{1}{x(1+x)}]$$

$$\lim_{x\to 0} -(1+x)^{\frac{1}{x}} = -e$$

$$\begin{aligned}\lim_{x\to 0} -\frac{\ln(1+x)}{x^2}+\frac{1}{x(1+x)} &= \lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\\ &= \lim_{x\to 0}\frac{-\ln(1+x)}{x^2+2x(1+x)}\\ &= \lim_{x\to 0}\frac{-\frac{1}{1+x}}{6x+2}\\ &= -\frac{1}{2}\end{aligned}$$

Hence $\lim_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{x} = \dfrac{e}{2}$

(k)

$$\begin{aligned}\lim_{x\to 0}\frac{2^x-1}{x} &= \lim_{x\to 0}\frac{2^x\ln 2}{1}\\ &= \ln 2\end{aligned}$$

(l) Let $y = x^{\frac{1}{1+\ln x}}$

$$\begin{aligned}\ln y &= \frac{\ln x}{1+\ln x}\\ \lim_{x\to 0^+}\ln y &= \lim_{x\to 0^+}\frac{\ln x}{1+\ln x}\\ &= 1\\ \ln(\lim_{x\to 0^+} y) &= 1\\ \lim_{x\to 0^+} y &= e\end{aligned}$$

(m) Let $y = x^{\frac{1}{1-x}}$

$$\begin{aligned}
\ln y &= \frac{\ln x}{1-x} \\
\ln \lim_{x\to 1} y &= \lim_{x\to 1} \frac{\ln x}{1-x} \\
&= \lim_{x\to 1} \frac{\frac{1}{x}}{-1} \\
\lim_{x\to 1} y &= e^{-1}
\end{aligned}$$

(n)

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\ln(2x^3-5x^2+3)}{\ln(4x^2+x-7)} &= \lim_{x\to+\infty} \frac{6x^2-10x}{2x^3-5x^2+3} \cdot \frac{4x^2+x-7}{8x+1} \\
&= \frac{3}{2}
\end{aligned}$$

(o)

$$\begin{aligned}
\lim_{x\to+\infty} x\sin(\frac{1}{x}) &= \lim_{x\to+\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \\
&= 1
\end{aligned}$$

(p)

$$\begin{aligned}
\lim_{x\to+\infty} x(\frac{\pi}{2} - \tan^{-1} x) &= \lim_{x\to+\infty} \frac{(\frac{\pi}{2} - \tan^{-1} x)}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= 1
\end{aligned}$$

(q)

$$\begin{aligned}\lim_{x\to+\infty} x\ln(1+\frac{3}{x}) &= \lim_{x\to+\infty} \ln(1+\frac{3}{x})^x \\ &= \ln(\lim_{x\to+\infty}(1+\frac{3}{x})^x) \\ &= 3\end{aligned}$$

(r) Let $y=(e^x+x)^{\frac{1}{x}}$

$$\begin{aligned}\lim_{x\to+\infty} \ln y &= \lim_{x\to+\infty} \frac{\ln(e^x+x)}{x} \\ &= \lim_{x\to+\infty} \frac{e^x}{e^x+x} \\ &= 1\end{aligned}$$

$$\lim_{x\to+\infty} y = e$$

(s)

$$\begin{aligned}\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{x}}}{x^4} &= \lim_{x\to 0} \frac{-\sin x + xe^{-\frac{x^2}{2}}}{4x^3} \\ &= \lim_{x\to 0} \frac{-\cos x + e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}}{12x^2} \\ &= \lim_{x\to 0} \frac{\sin x - 3xe^{-\frac{x^2}{2}} + x^3e^{-\frac{x^2}{2}}}{24x} \\ &= \lim_{x\to 0} \frac{\cos x - 3e^{-\frac{x^2}{2}} + 6x^2e^{-\frac{x^2}{2}} - x^4e^{-\frac{x^2}{2}}}{24} \\ &= -\frac{1}{12}\end{aligned}$$

(t)

$$\begin{aligned}\lim_{x\to 0} \frac{e^x\sin x - x(1+x)}{x^3} &= \lim_{x\to 0} \frac{3^x\sin x + e^x\cos x - 2x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{2e^x\cos x - 2}{6x} \\ &= \lim_{x\to 0} \frac{2e^x\cos x - 2e^x\sin x}{6}\end{aligned}$$

(u)

$$\begin{aligned}
&\lim_{x\to+\infty} x^{\frac{3}{2}}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})\\
&=\lim_{x\to+\infty} x^{\frac{3}{2}}\Big(\frac{1}{\sqrt{x+1}+\sqrt{x}}-\frac{1}{\sqrt{x-1}+\sqrt{x}}\Big)\\
&=x^{\frac{3}{2}}\frac{\sqrt{x-1}-\sqrt{x+1}}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x-1}+\sqrt{x})}\\
&=x^{\frac{3}{2}}\frac{-2}{(\sqrt{1+\frac{1}{x}}+1)(\sqrt{1-\frac{1}{x}}+1)(\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{1}{x}})}\\
&=-\frac{1}{4}
\end{aligned}$$

(v)

$$\begin{aligned}
\lim_{x\to+\infty}\Big[x-x^2\ln(1+\frac{1}{x}\Big] &= \lim_{x\to+\infty}\frac{1-x\ln(1+\frac{1}{x}}{\frac{1}{x}}\\
&=\lim_{x\to+\infty}\frac{-\ln(1+\frac{1}{x}+\frac{1}{x+1}}{-\frac{1}{x^2}}\\
&=\lim_{x\to+\infty}\frac{\frac{1}{x(x+1)}-\frac{1}{(x+1)^2}}{-2\frac{1}{x^3}}\\
&=\lim_{x\to+\infty}\frac{x^2}{2(x^2+2x+1)}\\
&=\frac{1}{2}
\end{aligned}$$

5. The degree-20 Taylor polynomials of the function f at point 0 is :

$$f(0)+\frac{f'(0)}{1!}x+\ldots+\frac{f^{(20)}(0)}{20!}x^{20}$$

And we have following result:

If f is infinitely differentiable at 0, and $g(x)=f(x^2)$

then g is infinitely differentiable at 0, with

$$g^{(m)}(0) = \begin{cases} 0 & \text{if m is odd} \\ \frac{(2k)!}{k!} f^{(k)}(0) & \text{if m is even and m=2k for some non-negative integer k} \end{cases}$$

(a) Let $f(x) = \sin x$ and $g(x) = f(x^2)$.

Then by the result, we have $g^{(2+4n)}(0) = (-1)^n \frac{(4n+2)!}{(2n+1)!}$ for $n = 0, 1, \ldots, 4$. Otherwise, $g^{(k)}(0) = 0$

Therefore, the required polynomials is $\frac{1}{1!}x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \frac{1}{9!}x^{18}$

(b) Let $f(x) = \sin(x^2)$ and $g(x) = f(x^2)$.

By the result, we have $g^{(4+8n)}(0) = (-1)^n \frac{(4+8n)!}{(1+2n)!}$ for $n = 0, 1, 2$. Otherwise, $g^{(k)}(0) = 0$

Therefore, the required polynomials is $\frac{1}{1!}x^4 - \frac{1}{3!}x^{12} + \frac{1}{5!}x^{20}$

(c) Let $f(x) = \sin(x^4)$ and $g(x) = f(x^2)$.

By the result, we have $g^{(8)}(0) = \frac{8!}{1!}$. Otherwise, $g^{(k)}(0) = 0$

Therefore, the required polynomials is x^8

(d) Let $f(x) = \cos x, g(x) = f(x^2)$ and $h(x) = g(x^2)$.

By the result, we have $g^{(4n)}(0) = (-1)^n \frac{(4n)!}{(2n)!}$ for $n = 1, 2, \ldots, 5, h^{(8)}(0) = -\frac{8!}{2!}, h^{(16)}(0) = \frac{16!}{4!}$. Otherwise, $g^{(k)}(0) = 0, h^{(k)}(0) = 0$

Therefore, the required polynomials of $\cos(x^2) - \cos(x^4)$ is $-\frac{1}{2!}x^4 + \frac{13}{4!}x^8 - \frac{1}{6!}x^{12} - \frac{1679}{8!}x^{16} - \frac{1}{10!}x^{20}$

(e) Since $\ln\left(\frac{1+x^4}{1+x^2}\right) = \ln(1 + x^4) - \ln(1 + x^2)$, we consider the Taylor polynomials of $\ln(1 + x^4)$ and $\ln(1 + x^2)$

Let $f(x) = \ln(1+x), g(x) = f(x^2)$ and $h(x) = g(x^2)$ By the result, we have $g^{(2n)}(0) = (-1)^{n+1} \frac{(2n)!}{n}$ for $n = 1, 2, \ldots, 10, h^{(4n)}(0) =$

$(-1)^{n+1}\frac{(4n)!}{n}$ for $n = 1, \ldots, 5$. Otherwise, $g^{(k)}(0) = 0, h^{(k)}(0) = 0$
Therefore, the required polynomials is $-x^2 + \frac{3}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 - \frac{1}{5}x^{10} + \frac{1}{2}x^{12} - \frac{1}{7}x^{14} - \frac{1}{8}x^{16} - \frac{1}{9}x^{18} + \frac{3}{10}x^{20}$